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# Some aspects of a finite $T_0$-$G$-space

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## 1 Introduction

The purpose of our presentation was to study actions of finite groups on finite $T_0$-spaces, i.e. topological spaces having finitely many points with the $T_0$-separation axioms. The definition of $T_0$-separation axiom is, for each pair of distinct points, there exists an open set containing one but not the other. A remarkable feature of a finite $T_0$-space is that it has the structure of a poset. Conversely, one can give any finite poset the structure of a finite $T_0$-space. The equvariant theory of finite $T_0$-spaces was first made by Stong [11]. After that, Kono and Ushitaki investigated the homeomorphism groups of finite spaces with group actions ([6], [7], [8]). Here a finite space is a topological space having finitely many points. In particular, they studied the homeomorphism groups of fixed point set $X^G$ and $G$-actions on homeomorphism groups induced by given $G$-action on $X$, where $X$ is a finite space with a G-action.

First we define a simplicail complex induced from a finite $T_0$-space. Recall that a finite $T_0$-space has a poset structure (see Proposition 2.2). Let $X$ be a finite poset. The *order complex* $\Delta(X)$ of $X$ is the abstract simplicial complex on the vertex set $X$ whose faces are the chains of $X$, including the empty chain. The *dimension* of a simplex is defined to be the length of the chain, where the length of a chain is one less than its number of elements. In particular, the length of the empty chain is $-1$. When the dimension of a simplex $\sigma$ is $k$, we write $\dim\sigma = k$. Next we shall define the geometric realization $|\Delta(X)|$ of $\Delta(X)$ by

$$|\Delta(X)| = \{m : X \to [0,1] \mid \sum_{x\in X} m(x) = 1,\ \mathrm{supp}(m) \in \Delta(X)\},$$

where for a map $m : X \to [0,1]$, we mean that $\mathrm{supp}(m) = \{x \in X \mid m(x) > 0\}$. The numbers $(m(x) \mid x \in X)$ are the *barycentric coordinates* of $m$. For a simplex $\sigma \in \Delta(X)$, we put

$$|\sigma| = \{m \in |\Delta(X)| \mid \mathrm{supp}(m) = \sigma\}.$$

We can define a metric topology on $|\Delta(X)|$. In details, we have a metric $d$ on $|\Delta(X)|$ defined by

$$d(m_1, m_2) = \left(\sum_{x\in X} (m_1(x) - m_2(x))^2\right)^{\frac{1}{2}}.$$

Then we have $\overline{|\sigma|} = \{m \in |\Delta(X)| \mid \sum_{x\in\sigma} m(x) = 1\}$, where $\overline{|\sigma|}$ indicates the closure of $|\sigma|$. Moreover a metric space $|\Delta(X)|$ is equipped with a $CW$-complex structure whose $n$-cell is a set $\{|\sigma| \mid \sigma \in \Delta(X),\ \dim\sigma = n\}$. Let $(p_x \mid x \in X)$ be a family of points in euclidean $n$-space $\mathbb{R}^n$. Consider the continuous map

$$f : |\Delta(X)| \to \mathbb{R}^n, \qquad m \mapsto \sum_{x \in X} m(x)p_x.$$

If $f$ is an embedding, we call the image of $f$ a *simplicial polyhedron* in $\mathbb{R}^n$ of type $\Delta(X)$, that is, $f(|\Delta(X)|)$ is a realization of $\Delta(X)$ as a polyhedron in $\mathbb{R}^n$.

Now, we shall introduce McCord's result [9, Theorem 2], which provides insight into understanding relations between finite $T_0$-spaces and simplicial complexes.

**Proposition 1.1.** *There exists a correspondence that assigns to each finite $T_0$-space $X$ a finite simplicial complex $\Delta(X)$, whose vertices are the points of $X$, such that the map $\mu_X : |\Delta(X)| \to X$ induced from the correspondence above is a weak homotopy equivalence. Moreover, each map $\varphi : X \to Y$ of finite $T_0$-spaces is also a simplicial map $\Delta(X) \to \Delta(Y)$, and $\varphi\mu_X = \mu_Y|\varphi|$ where $|\varphi| : |\Delta(X)| \to |\Delta(Y)|$ is a continuous map induced by $\varphi$.*

Let $G$ be a finite group. In this note, we focus on the equivariant order complex $\Delta(X)$ of a finite $T_0$-$G$-space $X$, that is, a finite $T_0$-space with a $G$-action, and then its orbit space $\Delta(X)/G$. In particular, we are interested in the following questions:

(i) Does $|\Delta(X)|$ has a $G$-$CW$-complex structure?
(ii) Is there the orbit space version of Proposition 1.1?

Our results related the above questions are the following.

**Theorem A.** Let $X$ be a finite $T_0$-$G$-space. Then $|\Delta(X)|$ is a finite $G$-$CW$-complex.

We will prepare the following technical condition:

(C) If $g_0, g_1, \cdots, g_k$ are elements of $G$ and $(x_0, x_1, \cdots, x_k)$ and $(g_0x_0, g_1x_1, \cdots, g_kx_k)$ are both simplices of $K$, then there exists an element $g$ of $G$ such that $gx_i = g_ix_i$ for all $i$. Here overlaps of some of $x_i$ are allowed.

**Theorem B.** If $\Delta(X)$ satisfies property (C), there exists a weak homotopy equivalence $\tilde{\mu}_X : |\Delta(X)|/G \to X/G$.

The rest of this note is organized as follows. In section 2, we briefly review finite ($T_0$-)space theory. In section 3, we investigate an equivariant version of finite $T_0$-spaces and prove Theorem A. The last section studies orbit spaces of equivariant complexes and prove Theorem B.

# 2 Finite ($T_0$-)spaces

In this section, we survey well-known properties about finite ($T_0$-)spaces. General reference may be found in [2], [6] and [10]. Let $X$ denote a finite space, i.e. a topological space having finitely many points. Let a set $U_x$ be the minimal open set which contains a point $x$ of $X$, that is, $U_x$ is the intersection of all open sets containing $x$. It is easy to see that a set $\{U_x\}_{x \in X}$ constitute a basis for the topology of $X$. Now we can define a *preorder* on $X$ by

$$x \leq y \quad \text{if} \quad x \in U_y.$$

In other words, every open set containing $y$ also contains $x$ if and only if $x \leq y$.

**Proposition 2.1.** *Let $x$ and $y$ be elements of a finite space $X$. Then $X$ is $T_0$-space if and only if $U_x = U_y$ implies $x = y$.*

**Proposition 2.2.** *A finite $T_0$-space with the above preorder $\leq$ is a poset.*

If $X$ is now a finite preordered set, one can define a topology on $X$ given by the basis $\{y \in X \,|\, y \leq x\}_{x \in X}$. Note that if $y \leq x$, then $y$ is contained in every basic set containing $x$, and therefore $y \in U_x$. Conversely, if $y \in U_x$, then $y \in \{z \in X \,|\, z \leq x\}$. After all, $y \leq x$ if and only if $y \in U_x$. This shows that these two applications, relating topologies and preorders on a finite set, are mutually inverse. Thus we have

**Proposition 2.3.** *A finite $T_0$-space corresponds to a finite poset.*

**Example 2.4.** Let $X = \{a, b, c\}$ be a finite space whose topology is $\{\emptyset, \{a, b, c\}, \{b, c\}, \{b\}, \{c\}\}$. This space is $T_0$. Immediately, $U_a = \{a, b, c\}$, $U_b = \{b\}$ and $U_c = \{c\}$. Therefore $b \leq a$ and $c \leq a$, but there exists no order relation between $b$ and $c$.

**Example 2.5.** Let $X = \{a, b, c, d\}$ be a finite space whose topology is $\{\emptyset, \{a, b, c, d\}, \{b, c, d\}, \{b\}, \{b, c\}, \{b, d\}\}$. This space is also $T_0$. Immediately, $U_a = \{a, b, c, d\}$, $U_b = \{b\}, U_c = \{b, c\}$ and $U_d = \{b, d\}$. On the order relation, we see the following Hasse diagram:

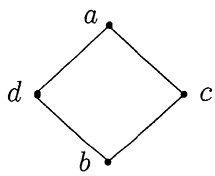



Figure 1.

**Proposition 2.6.** *Let $X$ be a preordered set. A set $F_x = \{y \in X \,|\, x \leq y\}$ is a closed set of $X$. Moreover $F_x$ is the closure of the set $\{x\}$.*

**Definition 2.7.** A subset $U$ of a preordered set $X$ is a *down-set* if for every $x \in U$ and $y \leq x$, it holds that $y \in U$. Dually, a subset $F$ of a preordered set $X$ is a *up-set* if for every $x \in F$ and $y \geq x$, it holds that $y \in F$. Open sets of finite spaces correspond to down-sets and closed sets to up-sets.

**Proposition 2.8.** *Let $X$ and $Y$ be finite spaces, and $f$ be a map from $X$ to $Y$. Then $f$ is continuous if and only if $f$ is an order-preserving map.*

**Proposition 2.9.** *Let $X$ be a finite space, $f$ a continuous map of $X$ into itself. If $f$ is either one-to-one or onto, then it is a homeomorphism.*

Next we state connectivity. First, for each $U_x$, we let $U_x \subset A \cup B$, where $A$ and $B$ are open sets of a finite space $X$. Then $x$ is in one set, say $x \in A$, immediately $U_x \subset A$. Thus any finite space is locally connected.

**Proposition 2.10.** *Let $x, y$ be two comparable points of a finite space $X$ and $x \leq y$. Then there exists a path from $x$ to $y$ in $X$, that is, a map $\alpha$ from the unit interval $I$ to $X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.*

Let $X$ be a finite preordered set. A *fence* in $X$ is a sequence $x_0, x_1, \cdots, x_n$ of points such that any two consecutive are comparable. $X$ is *order-connected* if any two points $x, y \in X$ there exists a fence starting in $x$ and ending in $y$.

**Proposition 2.11.** *Let $X$ be a finite space. Then the following are equivalent:*
*(i) $X$ is a connected topological space.*
*(ii) $X$ is an order-connected preordered set.*
*(iii) $X$ is a path-connected topological space.*

If $X$ and $Y$ are finite spaces, we can consider the finite set $Y^X$ of continuous maps from $X$ to $Y$ with the pointwise order: $f \leq g$ if $f(x) \leq g(x)$ for every $x \in X$.

**Proposition 2.12.** *Let $X$ and $Y$ be two finite spaces. Then pointwise order on $Y^X$ corresponds to the compact-open topology.*

**Corollary 2.13.** *Let $f, g : X \to Y$ be two maps between finite spaces. Then $f \simeq g$ if and only if there is a fence $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$. Moreover, if $A \subset X$, then $f \simeq g$ rel $A$ if and only if there exists a fence $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$ such that $f_i|_A = f|_A$ for every $0 \leq i \leq n$.*

Any finite space is homotopy equivalent to a finite $T_0$-space.

**Proposition 2.14.** *Let $X$ be a finite space. Let $X_0$ be the quotient $X/\sim$ where $x \sim y$ if $x \leq y$ and $y \leq x$. Then $X_0$ is $T_0$ and the quotient map $q : X \to X_0$ is a homotopy equivalence.*

Therefore , when studying homotopy types of finite spaces, we can restrict our attention to finite $T_0$-spaces.

**Definition 2.15.** A point $x$ in a finite $T_0$-space $X$ is a *down beat point* if $x$ cover one and only one element of $X$. This is equivalent to saying that the set $\hat{U}_x = U_x \backslash \{x\}$ has a maximum. Dually, $x \in X$ is an *up beat point* if $x$ is covered by a unique element or equivalently if $\hat{F}_x = F_x \backslash \{x\}$ has a minimum, where $F_x$ denotes the closure of the set $\{x\}$. In any of these cases, we say that $x$ is a *beat point* of $X$.

**Proposition 2.16.** *Let $X$ be a finite $T_0$-space and let $x \in X$ be a beat point. Then $X \backslash \{x\}$ is a strong deformation retract of $X$.*

**Definition 2.17.** A finite $T_0$-space is a *minimal finite space* if it has no beat points. A *core* of a finite space $X$ is a strong deformation retract which is a minimal finite space.

**Proposition 2.18.** *Let $X$ be a minimal finite space. A map $f : X \to X$ is homotopic to the identity if and only if $f = 1_X$.*

Immediately, we have the following corollary.

**Corollary 2.19. (Classification Theorem)** *A homotopy equivalence between minimal finite spaces is a homeomorphism. In particular, the core of a finite space is unique up to homeomorphism and two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.*

By the Classification Theorem, a finite space is contractible if and only if its core is a point. In fact, a one-point finite space has a core of the one-point. Therefore any contractible finite space has a point which is a strong deformation retract.This property is false in general for non-finite spaces.

# 3 Finite $T_0$-$G$-spaces

In this section, we treat an equivariant version of finite $T_0$-spaces. Let $G$ be a topological group (a group, for short) and $X$ a finite $T_0$-space. A $G$-invariant subspace $A \subset X$ is an *equivariant strong deformation retract* if there is an equivariant retraction $r : X \to A$ such that $ir$ is homotopic to $1_X$ via a $G$-homotopy which is stationary at $A$. A finite $T_0$-space which is a $G$-space will be *a finite* $T_0$-$G$-*space.*

**Remark** If a topological group $G$ acts on a finite topological space effectively, then it must be a finite topological group [7, Proposition 3.9]. Therefore, from now on, we assume that $G$ is finite.

**Proposition 3.1.** *Let $X$ be a finite $T_0$-$G$-space. Then there exists a core of $X$ which is $G$-invariant and an equivariant strong deformation retract of $X$.*

**Proposition 3.2.** *A contractible finite $T_0$-$G$-space has a point which is fixed by the action of $G$.*

This proposition deduces Stong's result stated in introduction. Note that $A_p(G)$ is a finite $T_0$-$G$-space by conjugation. If $A_p(G)$ is contractible, $A_p(G)$ has exactly one point core which is $G$-invariant. Therefore $A_p(G)$ has a fixed point by the action of $G$. Consequently, $G$ has a non-trivial normal $p$-subgroup.

**Proposition 3.3.** *Let $X$ and $Y$ be finite $T_0$-$G$-spaces and let $f : X \to Y$ be a $G$-map which is a homotopy equivalence. Then $f$ is an equivariant homotopy equivalence.*

Let $X$ be a finite $T_0$-$G$-space and $x$, $y$ points of $X$. If $x \in U_y$, then $gx \in gU_y = U_{gy}$. Therefore a $G$-action on a finite $T_0$-space $X$ preserves the order. Thus $\Delta(X)$ is a $G$-simplicial complex (in short, $G$-complex). Let $\mathbb{N}_0$ be the union set of natural numbers $\{1, 2, 3, \cdots\}$ and $\{0\}$.

**Definition 3.4.** Let $G$ be a finite group. A $CW$-complex $Z$ with a $G$-action is called a *$G$-$CW$-complex* if it satisfies the following conditions:
(i) The $G$-action determines a cellular map, that is, for any $g \in G$, $gZ^i \subset Z^i$ for each $i \in \mathbb{N}_0$, where $Z^i$ denotes the union of cells of dimension $\leq i$ and is called the *$i$-skeleton* of $Z$.
(ii) If $g(e) = e$, then $g$ is trivial on $\bar{e}$, that is, $Z^g \supset \bar{e}$, where $\bar{e}$ is the closure of $e$.

*Proof of Theorem A.*

*Proof.* For $g \in G$ and $m \in |\Delta(X)|$, we define a map $g(m) : X \to [0, 1]$ by

$$(g(m))(x) := m(g^{-1}(x)) \quad \text{for } x \in X.$$

Then we have

$$\sum_{x \in X}(g(m))(x) = \sum_{x \in X} m(g^{-1}(x)) = \sum_{g^{-1}(x) \in X} m(g^{-1}(x)) = 1,$$

on the other hand,

$$\begin{aligned}\operatorname{supp}(g(m)) &= \{x \in X \mid (g(m))(x) > 0\}\\ &= \{x \in X \mid m(g^{-1}(x)) > 0\}\\ &= \{x \in X \mid g^{-1}(x) \in \operatorname{supp}(m)\}\\ &= g(\operatorname{supp}(m)) \in \Delta(X).\end{aligned}$$

Therefore we have that $g(m) \in |\Delta(X)|$. Thus we can define a isometric map $g : |\Delta(X)| \to |\Delta(X)|$. For each $\sigma \in \Delta(X)$, it holds that $g(|\sigma|) = |g(\sigma)|$. In particular, a map $g$ is a cellular map.

Let $g(|\sigma|) = |\sigma|$. Immediately, we have $g(\sigma) = \sigma$. Since $g$ is an automorphism between totally ordered sets, it is an identity map. Therefore $g^{-1} : \sigma \to \sigma$ is also an identity map. Let $m$ be any element of $\overline{|\sigma|}$.

Case $x \in \sigma$ : It follows that $(g(m))(x) = m(g^{-1}(x)) = m(x)$.

Case $x \in X \backslash \sigma$ : Since $g^{-1}(x) \in X \backslash g^{-1}(\sigma) = X \backslash \sigma$, we get that $(g(m))(x) = m(g^{-1}(x)) = 0 = m(x)$.

Therefore $g(m) = m$. Thus we obtain that $\overline{|\sigma|} \subset |\Delta(X)|^g$. □

Referring to [5, p.229], we now prepare the following technical properties concerning a $G$-complex $K$:

($P_1$) For any $g \in G$ and simplex $\sigma$ of $K$, $g$ leaves $\sigma \cap g\sigma$ pointwise fixed.

($P_2$) If $g_0, g_1, \cdots, g_k$ are elements of $G$ and $(x_0, x_1, \cdots, x_k)$ and $(g_0x_0, g_1x_1, \cdots, g_kx_k)$ are both simplices of $K$, then there exists an element $g$ of $G$ such that $gx_i = g_ix_i$ for all $i$. Here overlaps of some of $x_i$ are allowed.

($P_3$) Let $g$ be an element of $G$ and $\sigma$ a simplex of $K$. If $g(\sigma) = \sigma$, $g$ leaves $\sigma$ pointwise fixed.

**Proposition 3.5.** *It holds that* $(P_2) \Longrightarrow (P_1) \Longrightarrow (P_3)$.

**Proposition 3.6.** *Let $X$ be a finite $T_0$-$G$-space. Then a $G$-complex $\Delta(X)$ holds both property $(P_1)$ and property $(P_3)$.*

On a $G$-complex, we can see a geometric simplex as a cell. One immediate consequence of this observation is the following.

**Proposition 3.7.** *Let $|K|$ be the geometric realization of a $G$-complex $K$ with property $(P_3)$. Then $|K|$ is a $G$-$CW$-complex.*

The following result is an equivariant version of Proposition 1.1 in a sense.

**Proposition 3.8.** *Let $X$ be a finite $T_0$-$G$-space. For each subgroup $H$ of $G$, it holds that $\Delta(X^H) = \Delta(X)^H$ and the map $\mu_X^H : |\Delta(X)|^H \to X^H$ is a weak homotopy equivalence.*

# 4 Orbit spaces

Next we will devote the study of the orbit space of a $G$-complex.

**Proposition 4.1.** *Let $X$ be a finite $T_0$-$G$-space. Then the orbit space $X/G$ is a finite $T_0$-space.*

Let $X$ and $Y$ be finite sets, and $\mathcal{P}(X)$ the power set of $X$. A map $f : X \to Y$ induces a map $\mathcal{P}(X) \to \mathcal{P}(Y)$, which we denote also by $f$. Let $K$ be a simplicial complex such that $X$ is the set of vertices of $K$. Then it is easy to see that the image $f(K)$ becomes a simplicial complex such that $f(X)$ is the set of vertices of $f(K)$. We apply this observation to our situation.

Let $K$ be a $G$-complex and $X$ be the set of vertices of $K$. Concerning the induced $G$-action on $X$. we consider its orbit space $X/G$ and the orbit map $p : X \to X/G$. As observes above, $p$ induced a map $\mathcal{P}(X) \to \mathcal{P}(X/G)$, which we denote by $p$ as well and $p(K)$ becomes a simplicial complex such that $X/G$ is the set of vertices of $p(K)$. For $s \in K$, we denote $p(s)$ by $\overline{s}$.

Next we consider another kind of orbit space. Let $K$ be a $G$-complex. Denote by $K/G$ the orbit space of the $G$-action on $K$ and by $\pi : K \to K/G$ the orbit map. For $s \in K$, we denote $\pi(s)$ by $[s]$. Note that $K/G$ is not a simplicial complex in general and $K/G$ does not coincide with $p(K)$ in general.

**Proposition 4.2.** [5, **Lemma 5.10**] *Let $K$ be a $G$-complex satisfying property $(P_2)$ and $X$ be the set of vertices of $K$. Then the orbit space $K/G$ becomes a simplicial complex such that the set of vetrices $K/G$ is $X/G$ and $K/G$ is naturally isomorphic to $p(K)$. Moreover the orbit map $\pi : K \to K/G$ is a simplicial map preserving dimension of simplexes.*

**Corollary 4.3.** *If $K$ is a $G$-complex satisfying property $(P_2)$, $|K|/G$ is homeomorphic to $|K/G|$.*

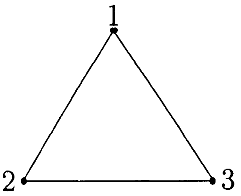
Furthermore, we add simplicial notion for both posets and (finite) cell complexes to investigate the simplicial structure of the orbit spaces in detail.

**Definition 4.4.** A *simplicial poset* $P$ is a finite poset with a smallest element $\hat{0}$ such that every interval

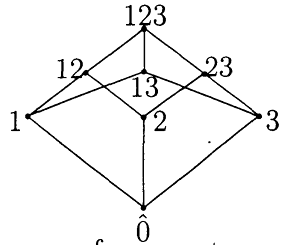
$$[\hat{0}, y] = \{x \in P \,|\, \hat{0} \leq x \leq y\}$$

for $y \in P$ is a boolean algebra, i.e., $[\hat{0}, y]$ is isomorphic to the set of all subsets of a finite set, ordered by inclusion. When a boolean algebra is the set of all subsets of a finite set consisting of $n$ elements, we denote the boolean algebra by $B_n$. Let $x$ be an element of $P$ such that $[\hat{0}, x]$ is isomorphic to a boolean algebra $B_n$. Then the *dimension* of $x$ is said to be $n-1$, denoted by $\dim x = n-1$. Remark that $\dim \hat{0} = -1$. Moreover, a simplicail poset $P$ is *$n$-dimensional*, if it contains at least one point $x$ such that $\dim x = n$ but no $(n+1)$-dimensional points.

The set of all faces of a (finite) simplicial complex with empty set added forms a simplicial poset ordered by inclusion, where the empty set is the smallest element. Such a simplicial poset is called the *face poset* of a simplicial complex, and two simplicial complexes are isomorphic if and only if their face posets are isomorphic. Therefore, a simplicial poset can be thought of as a generalization of a simplicial complex. Figure 2 shows that a 2-simplicial complex and its face poset.



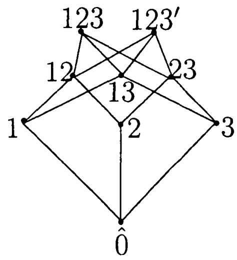
a 2-simplicial complex



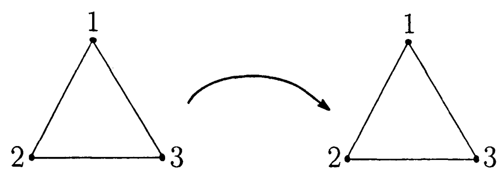
a face poset

Figure 2.

A $CW$-complex is said to be *regular* if all closed cells are homeomorphic to closed disks. Although a simplicial poset is not necessarilly the face poset of a simplicial complex, it is always the face poset of a regular $CW$-complex. Let $P$ be a simplicial poset. To each element $y \in P \backslash \{\hat{0}\} = \overline{P}$, we assign a (geometric) simplex whose face poset is $[\hat{0}, y]$ and glue those geometric simplices according to the order relation in $P$. Then, we get the $CW$-complex in which the closure of each cell is identified with a simplex, the structure of faces being preserved; moreover, all characteristic mappings are embeddings. This $CW$-complex is called a *simplicial cell complex* associated to $P$ and is denoted by $|P|$. For instance, if two 2-simplices are identified on their boundaries via the identity map, then it is not a simplicial complex but a $CW$-complex obtained from a simplicial poset (see Figure 3). Clearly, this $CW$-complex is homeomorphic to the 2-sphere $S^2$. The simplicial cell complex $|P|$ has a well-defined barycentric subdivision which is isomorphic to the order complex $\Delta(\overline{P})$ of the poset $\overline{P}$.



a simplicial poset $P$
(123′ is a copy of 123)



a simplicial cell complex $|P|$

Two 2-simplicial complexes are identified on their boundaries.

Figure 3.

By definition, we have the following proposition.

**Proposition 4.5.** *Let $S$ is a finite cell complex. Then $S$ is simplicial if and only if for each cell $\sigma \subset S$, the closure $\overline{\sigma}$ of $\sigma$ is isomorphic to a simplex $\Delta$ of the same dimension with $\sigma$ as a cell complex.*

In a word, a simplicial cell complex is a cell complex such that each closed cell is a geometric simplex. Obviously, the geometric realization of any finite simplicial complex is a simplicial cell complex.

**Definition 4.6.** Let $S$ be a simplicial cell complex and $V(S)$ the set of all 0-cells of $S$. Let $\sigma$ be a cell of $S$. We put $V(\sigma) = V(S) \cap \overline{\sigma}$. For each cell $\sigma \subset S$, there is an embedding

$$\varphi_\sigma : \Delta^{\dim \sigma}(V(\sigma)) \twoheadrightarrow \overline{\sigma} \subset S,$$

where $\Delta^{\dim \sigma}(V(\sigma))$ is the $\dim \sigma$-simplex whose vertex set is $V(\sigma)$. We say $\varphi_\sigma$ a *characteristic map* of $\sigma$.

**Proposition 4.7.** *A simplicial poset corresponds to a simplicial cell complex.*

Let $P$ be a simplicial poset and $x \in P$. A half-open interval $(\hat{0},\, x]$ is a subset $\{y \in P \,|\, \hat{0} \lneq y \leq x\}$ of $P$.

**Definition 4.8.** Let $P$ and $Q$ be simplicial posets. A *simplicial poset map* $f : P \to Q$ is a map such that for any $x \in P$, $\dim f(x) \leq \dim x$ and $f((\hat{0},\, x]) = (\hat{0},\, f(x)]$.

For a simplicial poset $P$, we put $V(P) := \{x \in P \,|\, \dim x = 0\}$, which is called the *vertex set* of $P$. Similarly, for each $x \in P$, $V(x) := V([\hat{0}, x]) = [\hat{0}, x] \cap V(P)$, which is also called the *vertex set* of $x$. A simplicial poset map $f$ is order-preserving and satisfies $f(V(x)) = V(f(x))$ for $x \in P$. Note that $V(P) = \bigcup_{x \in P} V(x)$. Moreover we put

$$K_P := \{V(x) \,|\, x \in P\},$$

which is a simplicial complex whose vertex set is $V(P)$. Here we see $K_P$ as a simplicial poset, so that a surjection $\varphi_P : P \to K_P$ defined by $\varphi_P(x) = V(x)$ is a simplicial poset map.

**Definition 4.9.** Let $X$ and $Y$ be simplicial cell complexes. A *simplicial cell complex map* $f : X \to Y$ is a cellular map such that for any cell $\sigma \in X$, $f(\sigma)$ is a cell of $Y$ and $f\,|_{\overline{\sigma}} : \overline{\sigma} \to \overline{f(\sigma)} \subset Y$ extends linearly the map $f\,|_{V(\sigma)} : V(\sigma) \to V(f(\sigma)) \subset Y$. Note that $f(\overline{\sigma})$ is the compact set of a Hausdorff space $Y$.

Let $X$ and $Y$ be simplicial cell complexes. Let $\mathcal{F}(X)$ $(respectively, \mathcal{F}(Y))$ be a simplicial poset corresponding to $X$ $(respectively, Y)$. A simplicial cell complex map $f : X \to Y$ defines a simplicial poset map $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ by $\sigma \mapsto f(\sigma)$ for each cell $\sigma \in X$. Conversely, we have the following.

**Proposition 4.10.** *For any simplicial poset map $\alpha : \mathcal{F}(X) \to \mathcal{F}(Y)$, there exists uniquely a simplicial cell complex map $f : X \to Y$ such that $\mathcal{F}(f) = \alpha$. In particular, if a simplicial poset map $\alpha : \mathcal{F}(X) \to \mathcal{F}(Y)$ is bijective, then $f$ is an isomorphism from $X$ to $Y$.*

**Proposition 4.11.** *For any simplicial poset $P$, there exists some simplicial cell complex $X$ with $\mathcal{F}(X) \cong P$.*

From the above two propositions, there is uniquely an isomorphism class $[X]$ such that $\mathcal{F}(X) \cong P$. Then a simplicial cell complex $X$ is said to be *a realization* of $P$, denoted by $|P|$ as well. Under this notation, we have a simplicail cell complex map $|\varphi_P| : |P| \to |K_P|$.

Let $K$ be a $G$-complex. Now, we shall investigate the structure of the orbit space $K/G$. Let $\sigma$ and $\tau$ be simplices of $K$. We define a partial ordering on $K/G$ as follows:

$$\pi(\tau) \leq \pi(\sigma) \text{ if and only if there exists an element } g \in G \text{ such that } g(\tau) \subset \sigma,$$

where the map $\pi : K \to K/G$ is the orbit map. Note that the orbit space $K/G$ has the minimum $\hat{0} = \pi(\emptyset)$. Moreover we denote the orbit map from $|K|$ to $|K|/G$ by $\pi$ as well.

**Proposition 4.12.** *If a $G$-complex $K$ has property $(P_1)$, $K/G$ is a simplicial poset. Moreover $|K|/G$ is a simplicial cell complex such that $\{\pi(|\sigma|) \mid \sigma \in K \backslash \{\emptyset\}\}$ is the set of all cells of $|K|/G$.*

**Proposition 4.13.** *If a $G$-complex $K$ has property $(P_1)$, it holds that $|K|/G \cong |K/G|$ as a simplicial cell complex.*

**Corollary 4.14.** *Let $X$ be a finite $T_0$-$G$-space. The orbit space $|\Delta(X)|/G$ is a finite simplicial cell complex associated to a simplicial poset $\Delta(X)/G$. Moreover we have $|\Delta(X)|/G \cong |\Delta(X)/G|$.*

Let $X$ be a finite $T_0$-$G$-space. Since the orbit map $p : X \to X/G$ is continuous, it is an order-preserving map. It determines a simplicial map

$$\Delta(p) : \Delta(X) \to \Delta(X/G),$$

and also a continuous map $|\Delta(p)| : |\Delta(X)| \to |\Delta(X/G)|$. Noting $|\Delta(X/G)|$ is a $G$-space with a trivial $G$-action, we have a continuous map $\tilde{p} : |\Delta(X)|/G \to |\Delta(X/G)|$ such that the following diagram commutes

$$\begin{tikzcd}
|\Delta(X)| \arrow[d, "q"'] \arrow[dr, "|\Delta(p)|"] & \\
|\Delta(X)|/G \arrow[r, "\tilde{p}"'] & |\Delta(X/G)|
\end{tikzcd}$$

where $q$ is the orbit map from $|\Delta(X)|$ to $|\Delta(X)|/G$.

**Proposition 4.15.** *Let $X$ be a finite $T_0$-$G$-space. A simplicial complex $K_{\Delta(X)/G}$ concides with $\Delta(X/G)$.*

In consequence we have the following commutative diagram:

$$\begin{tikzcd}
|\Delta(X)|/G \arrow[r, "\cong"] \arrow[d, "\tilde{p}"'] & |\Delta(X)/G| \arrow[d, "|\varphi_{\Delta(X)/G}|"] \\
|\Delta(X/G)| \arrow[r, "id"'] & |\Delta(X/G)|.
\end{tikzcd}$$

A simplicial action of $G$ on a simplicial complex $K$ is called *regular in the sense of Bredon* if $K$ possesses property ($\mathrm{P}_2$) for the action of each subgroups of $G$. Now, we shall present an interesting example.

**Example 4.16.** Let $n$ be an integer larger than one. Let $X_{2n+2}$ be a set consisting of $2n+2$ elements as follows:

$$X_{2n+2} =: \bigcup_{i=1}^{n+1} \{x_i,\, x_{-i}\}.$$

We set

$$\begin{cases} U(x_i) := \{x_i\} \bigcup\limits_{j=1}^{i-1} \{x_j,\, x_{-j}\}, & \text{and} \\ U(x_{-i}) := \{x_{-i}\} \bigcup\limits_{j=1}^{i-1} \{x_j,\, x_{-j}\}, \end{cases}$$

for $i = 1, 2, \cdots, n+1$. First note that each point $x_i$ determines the smallest open set $U(x_i)$ on $X_{2n+2}$, that is, $U_{x_i} = U(x_i)$. Therefore we define a $T_0$-topology on $X_{2n+2}$. Let $g$ be a map from $X_{2n+2}$ to itself by $g(x_i) = x_{-i}$. We set $G := \langle g \rangle$(that is, a group is generated by $g$). Evidently, $G$ is a cyclic group whose order is two. Since $|\Delta(X_{2n+2})|$ is homeomorphic to the $n$-sphere $S^n$, it holds that $|\Delta(X_{2n+2})|/G \cong \mathbb{R}P^n$, where $\mathbb{R}P^n$ is the $n$-dimensional real projective space. Note that $|\Delta(X_{2n+2})|/G$ is a simplicial cell complex by Proposition 4.12. On the other hand, $X_{2n+2}/G$ is a totally ordered set with $n+1$ elements. Therefore $|\Delta(X_{2n+2}/G)|$ is homeomorphic to a $n$-simplex $\Delta^n(X_{2n+2}/G)$. Since the map $\tilde{p} : |\Delta(X_{2n+2})|/G \to |\Delta(X_{2n+2}/G)|$ is not a weak homotopy equivalence, $\tilde{p}$ is not an isomorphism between simplicial cell complexes. If $\Delta(X_{2n+2})/G$ is a simplicial complex, the map $|\varphi_{\Delta(X_{2n+2})/G}|$ is an isomorphism, and $\tilde{p}$ is also an isomorphism. This is a contradiction. Hence $\Delta(X_{2n+2})/G$ is not a simplicial complex, thereby $G$-action on $\Delta(X_{2n+2})$ is not regular in the sense of Bredon.

*Proof of Theorem B.*

Let $X$ be a finite $T_0$-$G$-space. By Proposition 1.1, there is a weak homotopy equivalence $\mu_X : |\Delta(X)| \to X$. Then $\mu_X$ determines a continuous map $\tilde{\mu}_X : |\Delta(X)|/G \to X/G$ such that the following diagram commutes.

$$\begin{tikzcd} {|\Delta(X)|/G} \arrow[r, "\tilde{p}"] \arrow[dr, "\tilde{\mu}_X"'] & {|\Delta(X/G)|} \arrow[d, "\mu_{X/G}"] \\ & X/G \end{tikzcd}$$

Therefore $\tilde{p}$ is a weak homotopy equivalence if and only if $\tilde{\mu}_X$ is so. In general, $\tilde{\mu}_X$ is not a weak homotopy equivalence (see Example 4.16).

Remark that both $|\Delta(X)|/G$ and $|\Delta(X/G)|$ are CW-complexes. Therefore, we have
**Claim 1.** $\tilde{\mu}_X$ is a weak homotopy equivalence if and only if $\tilde{p}$ is a homotopy equivalence.

We consider the case where $\tilde{p}$ is a homeomorphism.

**Claim 2.** Let $X$ be a finite $T_0$-$G$-space. Then the following conditions are equivalent:
(1) $\tilde{p}$ is a homeomorphism.
(2) $\Delta(X)/G$ is a simplicial complex.
(3) $\Delta(X)$ has property ($P_2$).

*Proof.* (1) $\Longrightarrow$ (2) Since $\tilde{p}$ is a homeomorphism, $\varphi_{\Delta(X)/G}$ is injective. Let $U$ be a subset of $X/G$. Then there exists only one element $s$ of $\Delta(X)/G$ at most with $V(s) = U$. Therefore $\Delta(X)/G$ is a simplicial complex. (2) $\Longrightarrow$ (1) Since $\Delta(X)/G$ is a simplicial complex, it holds that $|\Delta(X)/G| = |\Delta(X/G)|$. Noting that $\varphi_{\Delta(X)/G}$ is surjective, $\tilde{p}$ is also surjective.

By Proposition 2.9, $\tilde{p}$ is a homeomorphism. (2) $\Longrightarrow$ (3) Let $\sigma = \{x_i \,|\, i = 0, \cdots, k\}$ and $\tau = \{g_i x_i \mid g_i \in G,\ i = 0, \cdots, k\}$ be simplices of $\Delta(X)$. If $x_i = x_j$, then

$$g_j x_j = (g_j g_i^{-1})(g_i x_i) \in \tau \cap (g_j g_i^{-1})\tau.$$

Since a $G$-complex $\Delta(X)$ has property ($\mathrm{P}_1$), we have $g_j x_j = (g_j g_i^{-1})^{-1}(g_j x_j) = g_i x_j$, so that $g_i x_i = g_i x_j = g_j x_j$. Hence we assume that each $x_i\,(i = 0, \cdots, k)$ is distinct, then both $\sigma$ and $\tau$ are $k$-simplices of $\Delta(X)$. Therefore both $\pi(\sigma)$ and $\pi(\tau)$ are elements of $\Delta(X)/G$ such that $V(\pi(\sigma)) = V(\pi(\tau)) = \{\pi(x_i) \,|\, i = 0, \cdots, k\}$. By assumption, $\pi(\sigma) = \pi(\tau)$. In consequence there is some $g \in G$ such that $\tau = g(\sigma)$ and $g_i x_i = g x_i\,(i = 0, \cdots, k)$. (3) $\Longrightarrow$ (2) It follows from Proposition 4.2. □

Combining Claim 1 and Claim 2, we obtain Theorem B. □

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